

Determining the twist in an optical fiber

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Abstract

We determine the twist in a birefringent optical fiber from measurements, at one end of the fiber, of the fiber response to an impulsive source at the same end. This is the inverse problem of determining a non-constant coefficient, of a first order hyperbolic system in one space dimension with two speeds of propagation, from measurements at one end of an interval, of the solution of this system corresponding to an impulsive source at the same end. We prove a stability result for this inverse problem and give a provable reconstruction algorithm for this inverse problem.

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1 Introduction

We determine the twist in a birefringent optical fiber from measurements, at one end of the fiber, of the fiber response to an impulsive source at the same end.

Consider a birefringent fiber stretching along the z axis, with two channels with different but constant speeds of propagation twisting around each other with the twist captured by a real valued function $\beta(z)$ on $[0, \infty)$ with $\beta(0) = 0$, $\beta'(0) = 0$. The fiber is probed by an impulsive source from the left end, and the fiber response is measured at the same end (see Figure 1.1). The goal is to determine the twist $\beta(z)$ for $z > 0$ from the fiber response.

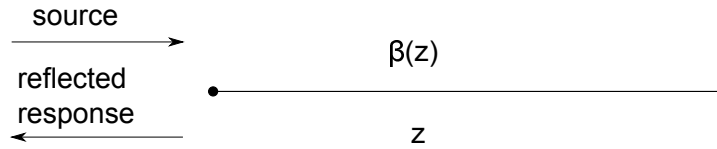


Figure 1.1: Fiber model

The forward problem was modeled in [10] and we reproduce this derivation in section 6 since it is not readily available. Without loss of generality, we assume the two channels have speeds c and 1 with $0 < c < 1$ and the four component vector function $\mathbf{M}(z, t)$ represents the signal at position z at time t with the M_1, M_3 components denoting the left moving waves of speeds 1 and c respectively, and M_2, M_4 components the right moving waves of speeds 1 and c (see Figure 1.2). The propagation and the reflection of the impulsive source in the

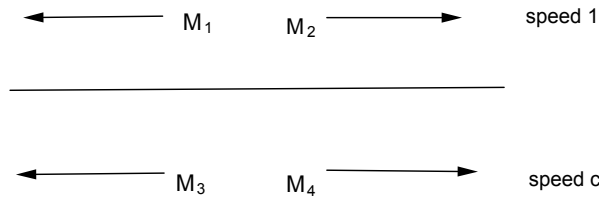


Figure 1.2: Left and right moving waves

twisted fiber is modeled by the solution of the initial boundary value problem (IBVP) for the hyperbolic system of PDEs

$$\mathbf{M}_t = A\mathbf{M}_z + \beta B\mathbf{M}, \quad z \geq 0, \quad t \in \mathbb{R}, \quad (1.1a)$$

$$M_2(0, t) = \delta(t), \quad M_4(0, t) = 0, \quad t \in \mathbb{R}, \quad (1.1b)$$

$$\mathbf{M}(z, t) = \mathbf{0}, \quad t < 0, \quad z \geq 0 \quad (1.1c)$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{bmatrix}, \quad B = \frac{1}{2} \begin{bmatrix} 0 & 0 & -1-c & -1+c \\ 0 & 0 & 1-c & 1+c \\ 1+c & -1+c & 0 & 0 \\ 1-c & -1-c & 0 & 0 \end{bmatrix}. \quad (1.2)$$

The well-posedness of the IBVP (1.1a) - (1.1c), for reasonably smooth $\beta(\cdot)$, is stated in Theorem 1.1. The initial and boundary conditions represent a plane wave sent from the left end of the fiber **along the faster channel** and our goal is to recover the twist $\beta(z)$ given the fiber response, $M_1(0, t)$ and $M_3(0, t)$, at the left end.

An analysis of the linearization of the map $\beta(\cdot) \rightarrow [M_1(0, \cdot), M_3(0, \cdot)]$, around $\beta = 0$, is instructive. Since the solution of (1.1a)-(1.1c) corresponding $\beta = 0$ is $[0, \delta(t - z), 0, 0]$, the linearization of the above map around $\beta = 0$ is the map

$$d\beta \rightarrow [dM_1(0, \cdot), dM_3(0, \cdot)]$$

where $d\mathbf{M}(z, t)$ is the solution of the IBVP

$$\begin{aligned} (d\mathbf{M})_t - A(d\mathbf{M})_z &= (d\beta)B[0, \delta(t - z), 0, 0], & (z, t) \in [0, \infty) \times \mathbb{R} \\ dM_2(0, t) &= 0, \quad dM_4(0, t) = 0, & t \in \mathbb{R} \\ d\mathbf{M}(z, t) &= 0, & t < 0, \quad z \in [0, \infty). \end{aligned}$$

Solving this IBVP one obtains that $dM_1(0, t) = 0$ and

$$(dM)_3(0, t) = \frac{c-1}{2(c+1)}(d\beta) \left(\frac{ct}{1+c} \right) H(t) \quad (1.3)$$

where $H(t)$ is the Heaviside function, so the linearization of the above mentioned map is

$$d\beta(z) \rightarrow [0, dM_3(0, t)]$$

with $dM_3(0, t)$ given by (1.3). The analysis of this linearized map suggests that, for the original (nonlinear) problem, to recover $\beta(z)$ on $[0, Z]$ one may need only $M_3(0, t)$ for all t in $[0, Z(1+c)/c]$.

Unfortunately, our results do not meet our expectations because our results require knowledge of both $M_1(0, t)$ and $M_3(0, t)$. Theorem 1.2 gives a stability result (and hence a uniqueness result) for the inverse problem and Theorem 1.3 asserts that we can reconstruct $\beta(\cdot)$ if we are given both $M_1(0, \cdot)$ and $M_3(0, \cdot)$ and an upper bound on the L^2 norm of β .

Below $l \preccurlyeq r$ will mean $l \leq Cr$ for some constant C , we define the operator

$$\mathcal{L} := I\partial_t - A\partial_z - \beta B$$

and, for any $Z > 0$, we define

$$\dot{C}^1[0, Z] = \{\beta \in C^1[0, Z] \mid \beta(0) = 0, \beta'(0) = 0\}, \quad Y = \frac{2cZ}{1+c}.$$

Our first result addresses the well-posedness of the IBVP (1.1a)-(1.1c).

Theorem 1.1 (Well-posedness). *If $\beta \in \dot{C}^1[0, Z]$ then (1.1a)-(1.1c) has a unique solution¹*

$$\mathbf{M}(z, t) = \delta(t - z)[0, 1, 0, 0] + \mathbf{m}(z, t)H(t - z)$$

where $\mathbf{m}(z, t)$ is the unique C^1 solution of the characteristic boundary value problem (CBVP)

$$\mathbf{m}_t = A\mathbf{m}_z + \beta B\mathbf{m}, \quad \text{on } 0 \leq z \leq t \leq 2Z - z, \quad (1.4a)$$

$$m_2(0, t) = m_4(0, t) = 0, \quad 0 \leq t \leq 2Z, \quad (1.4b)$$

$$m_1(z, z) = 0, \quad m_3(z, z) = \frac{c-1}{2(c+1)}\beta(z), \quad m_4(z, z) = \frac{c+1}{2(c-1)}\beta(z), \quad 0 \leq z \leq Z. \quad (1.4c)$$

Theorem 1.1 is valid only for those β with $\beta(0) = 0$, $\beta'(0) = 0$ - see the definition of $\dot{C}^1[0, Z]$, because these are forced² by the matching conditions if \mathbf{m} is to be C^1 .

The methods in this article can be modified to show that if $\beta \in L^2[0, Z]$ then the CBVP (1.4a)-(1.4c) has a weak solution which is locally L^2 on the region $t \geq z \geq 0$ and has local L^2 traces on lines parallel to the z or the t axes. This would be needed for a complete solution of our inverse problem but we do not prove this result here because we are unable to complete other parts of the solution of this inverse problem, as explained later.

Since $M_1(0, t)$ and $M_3(0, t)$ are zero for $t < 0$ and $M_1(0, t) = m_1(0, t)$, $M_3(0, t) = m_3(0, t)$ for $t \geq 0$, we will freely switch between $M_1(0, \cdot)$, $M_3(0, \cdot)$ and $m_1(0, \cdot)$, $m_3(0, \cdot)$ on the interval $[0, \infty)$.

Our next result shows that if the source is initiated in the fast channel then the reflected boundary data from both channels, over the time interval $[0, 2Z]$, is enough to stably distinguish the twist function $\beta(z)$, up to a depth Y . Note that, a signal originating at $z = 0$ at time $t = 0$, traveling at the fast speed 1, and reflected at $z = Y$ with the slower speed c , will just make it back to $z = 0$ at time $t = 2Z$ - see Figure 1.3. This theorem is relevant for the reconstruction of $\beta(\cdot)$ from the data. For arbitrary $K, Z > 0$ define

$$\mathcal{B}_K := \{\beta \in \dot{C}^1[0, Z] \mid \|\beta(\cdot)\|_{L^2[0, Z]}^2 \leq K\}.$$

¹on the region $\{(z, t) : 0 \leq z \leq Z, z + t \leq 2Z\}$ - see Figure 1.3

²Use (1.4a)-(1.4c) for m_3, m_4 . The condition $\beta(0) = 0$ is natural because it represents an untwisted fiber at the $z = 0$ end. The condition $\beta'(0) = 0$ is not natural and perhaps could be avoided if we work with β in the optimal regularity class, but that is unknown.

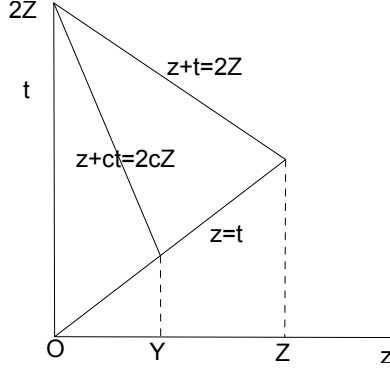


Figure 1.3: Depth sensed in time $2Z$

Theorem 1.2 (Injectivity and Stability). *If $\mathbf{M}, \tilde{\mathbf{M}}$ are the solutions of (1.1a)-(1.1c) corresponding to $\beta, \tilde{\beta} \in \mathcal{B}_K$ then*

$$\|(\beta - \tilde{\beta})(\cdot)\|_{L^2[0,Y]}^2 \preceq \|(M_1 - \tilde{M}_1)(0, \cdot)\|_{L^2[0,2Z]}^2 + \|(M_3 - \tilde{M}_3)(0, \cdot)\|_{L^2[0,2Z]}^2$$

where the constant depends only on c, Z and K .

Define the forward (nonlinear) map

$$\begin{aligned} \mathcal{F} : \dot{C}^1[0, Z] &\rightarrow C^1[0, 2Z] \times C^1[0, 2Z], \\ \beta(z) &\mapsto [m_1(0, t), m_3(0, t)] \end{aligned}$$

which maps the coefficient to the full reflection data. Theorem 1.2 guarantees that \mathcal{F} is injective and \mathcal{F}^{-1} is continuous in the appropriate norms, at least when β is restricted to the interval $[0, Y]$. Our main goal is to invert \mathcal{F} and we state our result in the following theorem. Again note that given $m_1(0, t), m_3(0, t)$ over $[0, 2Z]$, one recovers $\beta(\cdot)$ only on $[0, Y]$ and not on the whole interval $[0, Z]$.

Theorem 1.3 (Reconstruction). *If $\beta \in \dot{C}^1[0, Z]$ and $\mathbf{m}(z, t)$ is the corresponding solution of (1.4a)-(1.4c) then given $(m_1(0, t), m_3(0, t))$ for all $t \in [0, 2Z]$, one can reconstruct $\beta(\cdot)$ over the interval $[0, Y]$, if an upper bound on $\|\beta\|_{L^2[0,Y]}$ is also provided.*

Along with the inversion of \mathcal{F} , it is important to characterize the range of \mathcal{F} . Necessary conditions similar to those in [3] may be derived but they are far from sufficient for our problem. Actually $\dot{C}^1[0, Z]$ is not the appropriate domain for \mathcal{F} and the optimal answer will be obtained by studying the inversion and the range characterization of the map $\beta \rightarrow m_3(0, \cdot)$ rather than that of \mathcal{F} . We expect $L^2[0, Z]$ to be best suited for the domain of these maps. In our problem, the medium is probed by a source wave traveling at the faster speed - see the boundary conditions (1.1b). It would be interesting to also study the problem

when the source wave travels at the slower speed (the boundary conditions are changed to $M_2(0, t) = 0$, $M_4(0, t) = \delta(t)$). Unfortunately, we have no results for this case because of the complications due to the presence of precursor waves as noted by Belishev in his work; we will say more about this in the literature review next.

Inverse problems for hyperbolic PDE, in one space dimension, with a single speed of propagation, have been studied by Gelfand, Levitan, Marchenko, Krein, Blagoveschenski and many others; [6] and Browning's thesis [5] contain a thorough survey of these results. Inverse problems for hyperbolic systems, in one space dimension, with multiple speeds of propagation have been studied by Belishev and his collaborators (see [1], [3], [4] and specially [2] for an introduction to the method used by them), by Nizhnik and his collaborators (see [12]), and others; please see [13] for a brief survey. Inverse problems for hyperbolic PDEs with multiple speeds of propagation present challenges because of the presence of precursor waves. If the initial wave is an impulsive wave travelling with a slower speed, then an interaction with the medium (coefficients) may result in a smoother wave moving at a faster speed which reaches points in the medium before the more singular initial wave reaches there - this is the precursor wave. Since techniques used for inverse problems for single speed problems rely on the most singular wave arriving first or at the same time as the slower wave, new techniques need to be developed to solve the slower impulsive wave inverse problem. In [1], Belishev et al made an important observation and showed the way for solving inverse problems for multi-speed hyperbolic systems, which we describe next.

Define the diagonal matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix}$ with $0 < c < 1$ and let $P(z), Q(z)$ be arbitrary 2×2 matrices. For arbitrary $f_1(t), f_2(t)$, let $\mathbf{v}(z, t) \in \mathbb{R}^2$ be the solution of the two speed IBVP

$$\mathbf{v}_{tt} - D\mathbf{v}_{zz} - P\mathbf{v}_z - Q\mathbf{v} = 0, \quad (z, t) \in [0, \infty) \times \mathbb{R}, \quad (1.5a)$$

$$\mathbf{v} = 0, \quad t < 0, \quad (1.5b)$$

$$\mathbf{v}(0, t) = [f_1(t), f_2(t)]^T, \quad t \in \mathbb{R}. \quad (1.5c)$$

If f_1, f_2 are supported in the region $t \geq 0$ then, because of the finite speed of propagation, $\mathbf{v}(z, t)$ is supported in the fast region $t \geq z$. In [1], Belishev et al showed that there exists a unique $l(\cdot)$ such that \mathbf{v} is supported in the slow region $ct \geq z$ if $f_2 = l * f_1$, where $*$ represents convolution - see Figure 1.4. Then, in [1], [3], they considered the following inverse problem for a two speed hyperbolic system. Let $U(z, t)$ be the 2×2 matrix solution of the impulsive IBVP

$$U_{tt} - DU_{zz} - PU_z - QU = 0, \quad (z, t) \in [0, \infty) \times \mathbb{R}, \quad (1.6a)$$

$$U = 0, \quad t < 0, \quad (1.6b)$$

$$U(0, t) = \delta(t)I_2, \quad t \in \mathbb{R}; \quad (1.6c)$$

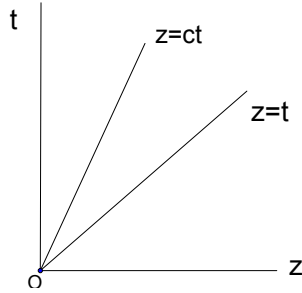


Figure 1.4: Fast and slow regions

here I_2 is the 2×2 identity matrix. Their goal was the recovery of the coefficients of $P(z)$ and $Q(z)$ over some interval $[0, Z]$, given $U_z(0, t)$ for all t in some interval $[0, T]$. The problem as stated is under-determined and is under-determined even if we assume the differential operator is self-adjoint - that is if the diagonal entries of P are zero and $Q - Q^T = P'$. Belishev et al showed that, in the self-adjoint case, one can recover $P(z)$ and $Q(z)$ if one is given $l(\cdot)$ in addition to $U_z(0, \cdot)$. They also had a data characterization result in [3] which is summarized in the introduction of [13].

For our problem, the goal is inversion without knowledge of $l(\cdot)$. In this direction, in [4], Belishev et al showed that if only $U_z(0, \cdot)$ is given (and $l(\cdot)$ is not given) and some of the coefficients of $P(z), Q(z)$ are known then $l(t)$ can be recovered over a small interval $[0, \delta]$ and hence the remaining coefficients of $P(z)$ and $Q(z)$ could be recovered over a small interval. This result was used by Morassi et al in [11] to prove a uniqueness result. Please see the introduction to [13] for a summary of these results. The recovery of $l(\cdot)$ over the full interval is an open question.

The article [13] also studies the recovery of P, Q from $U_z(0, \cdot)$ without knowledge of $l(t)$; a stability result is proved if some of the coefficients of P, Q are known but no reconstruction is provided. Please see the article for a careful statement.

Our work focuses on the reconstruction of a single coefficient of a two speed hyperbolic system without the knowledge of $l(t)$. We are given less data but we have to recover only one coefficient $\beta(z)$. We have borrowed ideas for inverse problems for single speed hyperbolic problems in [14], [15]. Normally this would fail for two speed problems for reasons pointed out above but due to the special structure of our problem we have succeeded in applying single speed ideas to our problem and proved Theorems 1.2 and 1.3.

This article is partly based on some of the work in the PhD thesis of Jiahua Tang.

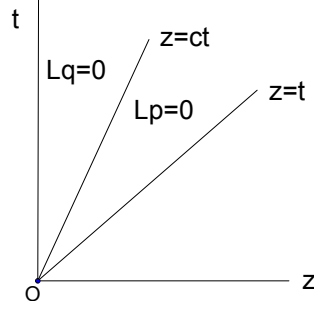


Figure 2.1: Subregions

2 Proof of Theorem 1.1

We show that the IBVP (1.1a)-(1.1c) is well posed. The solution $\mathbf{M}(z, t)$ is a distribution and it will be useful to express it in terms of standard distributions and well behaved functions. Using the progressing wave expansion and proceeding in a fashion similar to the derivation of Theorem 3 in [13], we can show that (see Figure 2.1),

$$\mathbf{M}(z, t) = \delta(t - z)[0, 1, 0, 0]^T + \mathbf{q}(z, t)H(t - z/c) + \mathbf{p}(z, t)(H(t - z) - H(t - z/c)) \quad (2.1)$$

where $\mathbf{p}(z, t), \mathbf{q}(z, t)$ is the solution of the characteristic transmission BVP

$$\mathcal{L}\mathbf{p} = \mathbf{0} \quad \text{on } 0 \leq ct \leq z \leq t, \quad (2.2a)$$

$$\mathcal{L}\mathbf{q} = \mathbf{0} \quad \text{on } 0 \leq z \leq ct, \quad (2.2b)$$

$$p_1(z, z) = 0, \quad p_3(z, z) = \frac{c-1}{2(1+c)}\beta(z), \quad p_4(z, z) = \frac{1+c}{2(c-1)}\beta(z), \quad z \geq 0, \quad (2.2c)$$

$$(q_1 - p_1)(z, t) = (q_2 - p_2)(z, t) = (q_3 - p_3)(z, t) = 0 \quad \text{on } z = ct, \quad z \geq 0, \quad (2.2d)$$

$$q_2(0, t) = q_4(0, t) = 0, \quad t \geq 0. \quad (2.2e)$$

For $Z > 0$, define (see Figure 2.2)

$$D_1 := \{(z, t) \mid ct \leq z \leq t, z + t \leq 2Z\},$$

$$D_2 := \{(z, t) \mid 0 \leq z \leq ct, z + t \leq 2Z\},$$

$$D := D_1 \cup D_2.$$

The well-posedness of (2.2a)-(2.2e) will follow from the well-posedness of the following

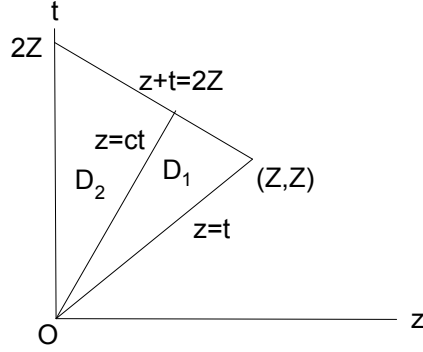


Figure 2.2: D_1 and D_2

general characteristic transmission BVP.

$$\mathcal{L}\mathbf{f} = \mathbf{0} \text{ in } D_1, \quad (2.3a)$$

$$\mathcal{L}\mathbf{g} = \mathbf{0} \text{ in } D_2, \quad (2.3b)$$

$$(g_i - f_i)(ct, t) = 0, \quad i = 1, 2, 3, \quad t \in \left[0, \frac{2Z}{1+c}\right], \quad (2.3c)$$

$$f_1(t, t) = b_1(t), \quad f_3(t, t) = b_3(t), \quad f_4(t, t) = b_4(t), \quad t \in [0, Z], \quad (2.3d)$$

$$g_2(0, t) = e_2(t), \quad g_4(0, t) = e_4(t), \quad t \in [0, 2Z]. \quad (2.3e)$$

One may verify that if $\mathbf{f} \cup \mathbf{g}$ is in $C^1(D)$ and $\beta(0) = 0$ then

$$b_4(0) = e_4(0), \quad 2(1-c)e_4'(0) = (1-c)b_1(0) - 2cb_4'(0) - (1+c)e_2(0). \quad (2.4)$$

We have the following result regarding the well-posedness of (2.3a)-(2.3e).

Proposition 2.1 (Existence of C^1 solutions). *If $\beta(\cdot) \in \dot{C}^1[0, 2Z]$, $b_i(t) \in C^1[0, Z]$, and $e_i(t) \in C^1[0, 2Z]$ and satisfy (2.4) then there exists a unique solution $\mathbf{f} \in C^1(D_1)$, $\mathbf{g} \in C^1(D_2)$ of (2.3a) – (2.3e) with $\|\mathbf{f}\|_{C^1}$, $\|\mathbf{g}\|_{C^1}$ bounded above by a function of c, Z and $N = \max(\|\beta\|_{C^0}, \|b_i\|_{C^1}, \|e_i\|_{C^1})$. Further $\mathbf{f} \cup \mathbf{g}$ is a C^1 function on D .*

Proof of Proposition 2.1. The existence of the solution will be reduced to the solution of an integral equation. Below $\mathbf{v}(z, t)$ will represent a 4 component vector function on $D_1 \cup D_2$ and $\mathbf{r}(\mathbf{v}, z, t) = \beta(z)B\mathbf{v}(z, t)$.

By integrating (2.3a)-(2.3b) along the characteristics and using the boundary conditions, we may show that the existence of a classical solution of (2.3a)-(2.3b) reduces to solving the

system of integral equations (see Figure 2.3)

$$v_1(z, t) = \int_{s_H}^t r_1(\mathbf{v}, z + t - s, s) ds + b_1(s_H), \quad \text{if } P \in D \quad (2.5a)$$

$$v_2(z, t) = \int_{s_E}^t r_2(\mathbf{v}, z + s - t, s) ds + e_2(s_E), \quad \text{if } P \in D \quad (2.5b)$$

$$v_3(z, t) = \int_{s_G}^t r_3(\mathbf{v}, z + ct - cs, s) ds + b_3(s_G), \quad \text{if } P \in D \quad (2.5c)$$

$$v_4(z, t) = \begin{cases} \int_{s_F}^t r_4(\mathbf{v}, z + cs - ct, s) ds + b_4(s_F), & \text{if } P \in D_1 \\ \int_{s_F}^t r_4(\mathbf{v}, z + cs - ct, s) ds + e_4(s_F), & \text{if } P \in D_2 \end{cases} \quad (2.5d)$$

where s_E, s_F, s_G, s_H are the s coordinates of the points E, F, G, H in the (y, s) plane in Figure 2.3.

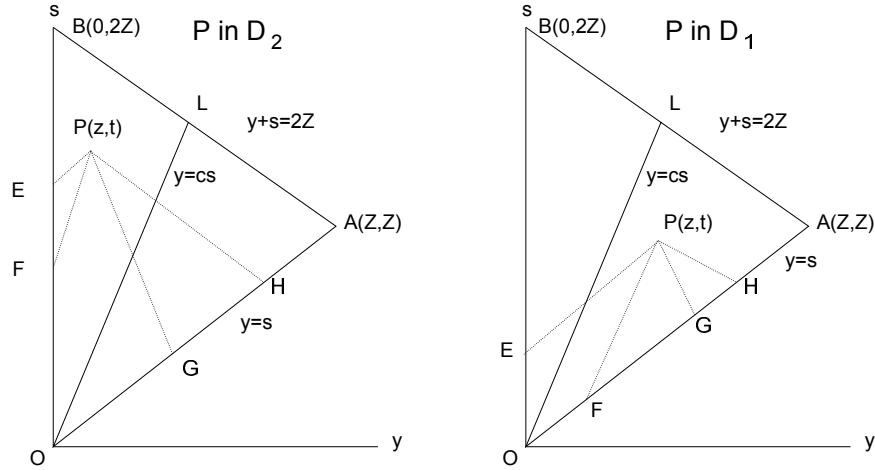


Figure 2.3: Downward moving lines through $P(z, t)$ with slopes ± 1 and $\pm 1/c$

If \mathbf{v} is in $C^1(D)$ then one may verify that the RHS of (2.5a)-(2.5d) is in $C^1(D)$ - the first order derivatives on $z = ct$ match as one approaches this line from the two different sides. Further, the system of integral equations is a Volterra type equation so the existence and uniqueness of a C^1 solution may be proved by standard arguments for Volterra equations or one may use the method in section 2.5 of [9]. \square

3 Proof of Theorem 1.2

We will use the following identity in several places in this article: for arbitrary four dimensional C^1 vector functions $\mathbf{u}(z, t), \mathbf{v}(z, t)$, since $B^T = -B$, we can verify that

$$\mathbf{u}^T(\mathcal{L}\mathbf{v}) + (\mathcal{L}\mathbf{u})^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})_t - (\mathbf{u}^T A\mathbf{v})_z. \quad (3.1)$$

The proofs of Theorem 1.2 and Theorem 1.3 will use sideways energy estimates given below.

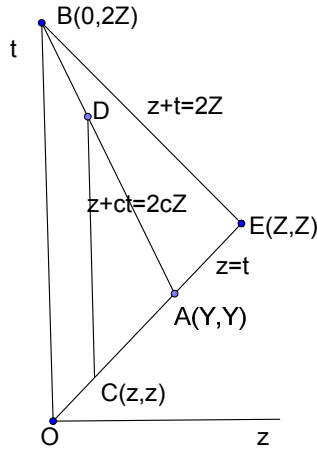


Figure 3.1: Regions used for the energy function

Define (the triangle OAB in Figure 3.1)

$$\tilde{D} := \{(z, t) \mid 0 \leq z \leq Y, z \leq t \leq 2Z - z/c\}$$

and for any 4 dimensional vector function $\mathbf{p}(z, t) \in C^1(\tilde{D})$, $\epsilon > 0$ define

$$\begin{aligned} \text{(sideways energy)} \quad J(\mathbf{p}, z) &:= \int_{CD} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2)(z, t) dt, \quad z \in [0, Y], \\ |\mathbf{p}(z, t)|^2 &:= \sum_{i=1}^4 p_i^2(z, t), \quad |(\mathcal{L}\mathbf{p})(z, t)|^2 := \sum_{i=1}^4 |(\mathcal{L}\mathbf{p})_i|^2(z, t), \quad (z, t) \in \tilde{D}. \end{aligned}$$

Lemma 3.1 (Sideways energy estimate). *If $\beta \in L^2[0, Y]$, $\mathbf{p}(z, t) \in C^1(\tilde{D})$, then for every $\lambda > 0$, $\epsilon \in (0, 1]$ and $z \in [0, Y]$ (see Figure 3.1)*

$$\begin{aligned} J(\mathbf{p}, z) &+ \int_{OC} (2p_1^2 + (1+c)p_3^2 - \epsilon(1-c)p_4^2)(y, y) dy \\ &\leq J(\mathbf{p}, 0) + \lambda \iint_{OCDB} |(\mathcal{L}\mathbf{p})(y, t)|^2 dx dt + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(y)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y) dy. \end{aligned} \quad (3.2)$$

Furthermore, if $\epsilon \leq \frac{c(1-c)^3}{(1+c)^4}$ and \mathbf{p} satisfies

$$\mathcal{L}\mathbf{p} = 0, \quad \text{in } \tilde{D}, \quad (3.3a)$$

$$p_3(z, z) = \frac{(c-1)^2}{(c+1)^2} p_4(z, z), \quad z \in [0, Y], \quad (3.3b)$$

then

$$J(\mathbf{p}, z) + \int_{OC} p_3^2 dy \leq e^{4\sqrt{Y\|\beta\|/(c\epsilon)}} J(\mathbf{p}, 0), \quad z \in [0, Y] \quad (3.4)$$

where $\|\beta\|$ is the L^2 norm of β on $[0, Y]$.

We postpone the proof of Lemma 3.1 to subsection 3.1 and continue with the proof of Theorem 1.2.

Let $\mathbf{m}, \tilde{\mathbf{m}}$ be the solutions of (1.4a) – (1.4c) corresponding to $\beta, \tilde{\beta}$ in \mathcal{B}_K . Since $\tilde{\mathcal{L}}\tilde{\mathbf{m}} = 0$ and $\tilde{\mathbf{m}}^T B \tilde{\mathbf{m}} = 0$ because $B^T = -B$, we have

$$\begin{aligned} 0 &= 2\tilde{\mathbf{m}}^T \tilde{\mathcal{L}}\tilde{\mathbf{m}} = 2\tilde{\mathbf{m}}^T \tilde{\mathbf{m}}_t - 2\tilde{\mathbf{m}}^T A \tilde{\mathbf{m}}_z - 2\tilde{\beta} \tilde{\mathbf{m}}^T B \tilde{\mathbf{m}} = 2\tilde{\mathbf{m}}^T \tilde{\mathbf{m}}_t - 2\tilde{\mathbf{m}}^T A \tilde{\mathbf{m}}_z \\ &= (\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2 + \tilde{m}_4^2)_t - (\tilde{m}_1^2 - \tilde{m}_2^2 + c\tilde{m}_3^2 - c\tilde{m}_4^2)_z. \end{aligned}$$

Integrating this over the triangular region OEB (see Figure 3.1) and noting (1.4b), we obtain

$$\int_{OE} 2\tilde{m}_1^2 + (1+c)\tilde{m}_3^2 + (1-c)\tilde{m}_4^2 dz = \int_{EB} 2\tilde{m}_2^2 + (1-c)\tilde{m}_3^2 + (1+c)\tilde{m}_4^2 dt + \int_{OB} \tilde{m}_1^2 + c\tilde{m}_3^2 dt.$$

Hence, using (1.4c), we have

$$J(\tilde{\mathbf{m}}, 0) \leq \int_{OE} 2\tilde{m}_1^2 + (1+c)\tilde{m}_3^2 + (1-c)\tilde{m}_4^2 dz = \frac{1+3c^2}{2(1-c^2)} \int_0^Z \tilde{\beta}(z)^2 dz \leq \frac{1+3c^2}{2(1-c^2)} K. \quad (3.5)$$

Next, applying Lemma 3.1 with \mathbf{p} replaced by $\tilde{\mathbf{m}}$, β replaced by $\tilde{\beta}$ and $\epsilon = c(1-c)^3/(1+c)^4$ - note that (3.3a), (3.3b) hold - from (3.4) and (3.5) we obtain

$$J(\tilde{\mathbf{m}}, z) \leq \frac{1+3c^2}{2(1-c^2)} K e^{4\sqrt{Y}K^{1/4}/(c\epsilon)} = C_0 \text{ (define)}. \quad (3.6)$$

Define $\mathbf{p} = \mathbf{m} - \tilde{\mathbf{m}}$; then \mathbf{p} satisfies $\mathcal{L}\mathbf{p} = (\beta - \tilde{\beta})B\tilde{\mathbf{m}}$ and, from (1.4c), we have

$$p_3(z, z) = \frac{(c-1)^2}{(c+1)^2} p_4(z, z) = \frac{c-1}{2(c+1)} (\beta - \tilde{\beta})(z, z), \quad z \in [0, Y]. \quad (3.7)$$

Choose $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$, then

$$\int_{OC} ((1+c)p_3^2 - \epsilon(1-c)p_4^2) dy = \int_{OC} p_3^2 dy = \frac{(1-c)^2}{4(1+c)^2} \int_{OC} (\beta - \tilde{\beta})^2(y) dy,$$

so from (3.2) in Lemma 3.1 and (3.6) we have

$$\begin{aligned} J(\mathbf{p}, z) + \frac{(1-c)^2}{4(1+c)^2} \int_0^z (\beta - \tilde{\beta})^2(y) dy \\ \leq J(\mathbf{p}, 0) + 4\lambda \int_0^z (\beta - \tilde{\beta})^2(y) J(\tilde{\mathbf{m}}, y) dy + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(z)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y) dy \\ \leq J(\mathbf{p}, 0) + 4C_0\lambda \int_0^z (\beta - \tilde{\beta})^2(y) dy + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(z)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y) dy. \end{aligned} \quad (3.8)$$

Choose $\lambda = \frac{(1-c)^2}{32C_0(1+c)^2}$, then from (3.8) we have

$$J(\mathbf{p}, z) + \frac{(1-c)^2}{8(1+c)^2} \int_0^z (\beta - \tilde{\beta})^2(y) dy \leq J(\mathbf{p}, 0) + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(z)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y) dy; \quad (3.9)$$

hence from Gronwall's inequality

$$\int_0^Y (\beta - \tilde{\beta})^2(y) dy \leq J(\mathbf{p}, 0) \leq \int_0^{2T} ((m_1 - \tilde{m}_1)^2 + (m_3 - \tilde{m}_3)^2)(0, t) dt,$$

with the constant dependent only on c, Z, K . This completes the proof of Theorem 1.2.

3.1 Proof of Lemma 3.1

Define $\mathbf{q} := [p_1, -p_2, p_3, -\epsilon p_4]^T$; multiplying both sides of $\mathcal{L}\mathbf{p} = \mathbf{p}_t - A\mathbf{p}_z - \beta B\mathbf{p}$ by $-2\mathbf{q}^T$, we have

$$\begin{aligned} -2\mathbf{q}^T(\mathcal{L}\mathbf{p} + \beta B\mathbf{p}) &= 2\mathbf{q}^T(A\mathbf{p}_z - \mathbf{p}_t) \\ &= (p_1^2 + p_2^2 + cp_3^2 + \epsilon cp_4^2)_z - (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2)_t. \end{aligned} \quad (3.10)$$

Integrating the RHS of (3.10) over $OCDB$, we have

$$\begin{aligned}
& \iint_{OCDB} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2)_z - (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2)_t \, dx \, dt \\
&= \int_{\partial OCDB} (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2) \, dz + \int_{\partial OCDB} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2) \, dt \\
&= \int_{OC} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2) \, dt + J(\mathbf{p}, z) - J(\mathbf{p}, 0) + \int_{DB} (p_1^2 + p_2^2 + cp_3^2 + c\epsilon p_4^2) \, dt \\
&\quad - \int_{DB} c(p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2) \, dt + \int_{OC} (p_1^2 - p_2^2 + p_3^2 - \epsilon p_4^2) \, dt \\
&= \int_{DB} ((1-c)p_1^2 + (1+c)p_2^2 + 2c\epsilon p_4^2) \, dt + \int_{OC} (2p_1^2 + (1+c)p_3^2 - \epsilon(1-c)p_4^2) \, dt \\
&\quad + J(\mathbf{p}, z) - J(\mathbf{p}, 0). \tag{3.11}
\end{aligned}$$

Also

$$\begin{aligned}
& \iint_{OCDB} \text{LHS of (3.10)} \, dx \, dt \\
&\leq \iint_{OCDB} \left(\frac{|\mathbf{p}(y, t)|^2}{\lambda} + \lambda |(\mathcal{L}\mathbf{p})(y, t)|^2 + 4|\beta(y)| \cdot |\mathbf{p}(y, t)|^2 \right) \, dx \, dt \\
&\leq \lambda \iint_{OCDB} |(\mathcal{L}\mathbf{p})(y, t)|^2 \, dx \, dt + \frac{1}{c\epsilon} \int_0^z \left(4|\beta(y)| + \frac{1}{\lambda} \right) J(\mathbf{p}, y) \, dy \tag{3.12}
\end{aligned}$$

for all $\lambda > 0$ and $\epsilon \in (0, 1]$, so (3.2) follows directly from (3.11) – (3.12).

If \mathbf{p} satisfies (3.3a) – (3.3b) and $\epsilon \leq \frac{c(1-c)^3}{(1+c)^4}$, then

$$\int_{OC} (2p_1^2 + (1+c)p_3^2 - \epsilon(1-c)p_4^2) \, dy \geq \int_{OC} p_3^2 \, dy.$$

In (3.2) using (3.3a) and letting $\lambda \rightarrow \infty$, we have

$$J(\mathbf{p}, z) + \int_{OC} p_3^2 \, dy \leq J(\mathbf{p}, 0) + \frac{4}{c\epsilon} \int_0^z |\beta(y)| J(\mathbf{p}, y) \, dy,$$

so (3.4) follows directly from Gronwall's inequality.

4 Proof of Theorem 1.3

For arbitrary $Z > 0$, $K > 0$, define $Y = \frac{2cZ}{1+c}$ and (see Figure 4.1)

$$\begin{aligned}\mathcal{B}_K &:= \{\beta \in \dot{C}^1[0, Z] \mid \|\beta(\cdot)\|_{L^2[0, Z]}^2 \leq K\}, \\ \Omega &:= OAB = \{(z, t) \mid z \geq 0, z \leq t, z + t \leq 2Z\}, \\ \tilde{\Omega} &:= OCB = \{(z, t) \mid 0 \leq z \leq Y, z \leq t \leq 2Z - z/c\}.\end{aligned}$$

Because of Theorem 1.1, our goal is to recover $\beta(\cdot)$ on $[0, Y]$ given $m_1(0, \cdot)$, $m_3(0, \cdot)$ on $[0, 2Z]$

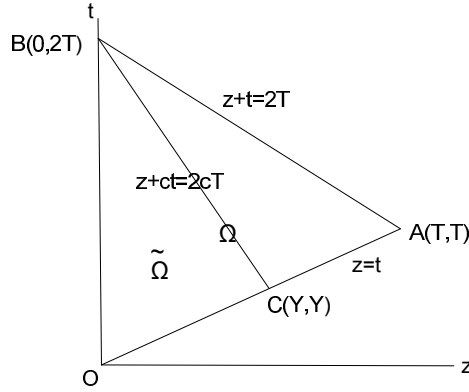


Figure 4.1: Ω and $\tilde{\Omega}$

where $\mathbf{m}(z, t)$ is the unique C^1 solution of (1.4a)-(1.4c). So our goal is the construction of the partial³ inverse (on the range) of the injective nonlinear map

$$\begin{aligned}\mathcal{F} : \dot{C}^1[0, Z] &\rightarrow C^1[0, 2Z] \times C^1[0, 2Z] \\ \beta(\cdot) &\rightarrow (m_1(0, \cdot), m_3(0, \cdot)).\end{aligned}$$

Fix an $(a_1(\cdot), a_3(\cdot))$ in the range of \mathcal{F} . For any $\beta(\cdot) \in \dot{C}^1[0, Y]$, consider the sideways problem

$$\mathcal{L}\mathbf{h} = \mathbf{0} \text{ in } \tilde{\Omega}, \tag{4.1a}$$

$$h_1(0, t) = a_1(t), \quad h_2(0, t) = 0, \quad h_3(0, t) = a_3(t), \quad h_4(0, t) = 0, \quad 0 \leq t \leq 2Z, \tag{4.1b}$$

$$h_3(z, z) = \frac{(c-1)^2}{(c+1)^2} h_4(z, z), \quad 0 \leq z \leq Y. \tag{4.1c}$$

³ because we recover $\beta(\cdot)$ only on $[0, Y]$

Note that if \mathbf{h} satisfies (1.4c) then \mathbf{h} satisfies (4.1c). We show that (4.1a)-(4.1c) has a unique C^1 solution for every $\beta(\cdot) \in \dot{C}^1[0, Y]$. But more importantly, we then show, **constructively**, that there is a unique $\beta(\cdot) \in \dot{C}^1[0, Y]$ such that

$$h_3(z, z) = \frac{(c-1)}{2(c+1)}\beta(z), \quad 0 \leq z \leq Y.$$

But we already know one such β . Since $(a_1(\cdot), a_3(\cdot))$ is in the range of \mathcal{F} , there is a β and an \mathbf{m} which solves (1.4a)-(1.4c) and such that $\mathcal{F}(\beta) = (a_1(\cdot), a_2(\cdot))$. Since this \mathbf{m} will also satisfy (4.1a)-(4.1c), the unique β found above must be the preimage of $(a_1(\cdot), a_3(\cdot))$ under \mathcal{F} .

Of the two claims mentioned in the previous paragraph, the first one about the well-posedness of the CBVP (4.1a)-(4.1c) will follow from a standard argument. The second claim, about a nonlinear problem, will be shown to be equivalent to the solution of a fixed point problem which will be studied by a contraction mapping argument. This will take some work because we will have to extend the idea of a solution of (4.1a)-(4.1c) to the case where $\beta \in L^2[0, Y]$ because our estimates will be L^2 estimates and hence to apply the contraction mapping theorem we will have to work square integrable β .

4.1 Well-posedness for the sideways CBVP

For this subsection, we drop the assumption that $\beta(0) = 0, \beta'(0) = 0$, for reasons which will become clear in the next subsection. For an arbitrary 4 dimensional vector function $\mathbf{a}(t) \in C^1[0, 2Z]$, consider the CBVP

$$\mathcal{L}\mathbf{h} = \mathbf{0} \text{ in } \tilde{\Omega}, \quad (4.2a)$$

$$h_3(z, z) = \frac{(c-1)^2}{(c+1)^2}h_4(z, z), \quad 0 \leq z \leq Y, \quad (4.2b)$$

$$\mathbf{h}(0, t) = \mathbf{a}(t), \quad 0 \leq t \leq 2Z. \quad (4.2c)$$

A simple but tedious calculation shows that the necessary (matching) conditions for (4.2a)-(4.2c) to have a C^1 solution are that

$$(c+1)^2 a_3(0) = (1-c)^2 a_4(0), \quad (4.3a)$$

$$(c+1)^2 ((1+c)a'_3(0) - \beta(0)(B\mathbf{a})_3(0)) = (c-1)^2 ((c-1)a'_4(0) + \beta(0)(B\mathbf{a})_4(0)). \quad (4.3b)$$

Proposition 4.1 (Well-posedness of the sideways CBVP). *If $\beta \in C^1[0, Y]$, and $\mathbf{a}(\cdot) \in C^1[0, 2Z]$ satisfies (4.3a)-(4.3b) then (4.2a)-(4.2c) has a unique solution $\mathbf{h} \in C^1(\tilde{\Omega})$.*

Proof. Define $\mathbf{r}(\mathbf{h}, z, t) := \beta(z)B\mathbf{h}(z, t)$. Integrating (4.2a) along the characteristics and using the boundary conditions (4.2b) – (4.2c), it is clear that proving Proposition 4.1 is equivalent to proving that the following system of integral equations has a unique C^1 solution (see Figure 4.2); here $P(z, t)$ is an arbitrary point in $\tilde{\Omega}$.

$$h_1(z, t) = \int_0^z r_1(\mathbf{h}, y, z + t - y) dy + a_1(s_C), \quad (4.4a)$$

$$h_2(z, t) = \int_0^z r_2(\mathbf{h}, y, y + t - z) dy + a_2(s_D), \quad (4.4b)$$

$$h_3(z, t) = \frac{1}{c} \int_0^z r_3\left(\mathbf{h}, y, \frac{z + ct - y}{c}\right) dy + a_3(s_E), \quad (4.4c)$$

$$h_4(z, t) = \begin{cases} \frac{1}{c} \int_0^z r_4\left(\mathbf{h}, y, \frac{y + ct - z}{c}\right) dy + a_4(s_F) & \text{if } z \leq ct \\ \frac{1}{c} \int_{y_H}^z r_4\left(\mathbf{h}, y, \frac{y + ct - z}{c}\right) dy + \frac{(1+c)^2}{(1-c)^2} \left(\frac{1}{c} \int_0^{y_H} r_3\left(\mathbf{h}, y, s_F - \frac{y}{c}\right) dy + a_3(s_F) \right) & \text{if } z \geq ct. \end{cases} \quad (4.4d)$$

Here s_E, s_C, y_H, \dots are the s, y coordinates of E, C, H, \dots in Figure 4.2.

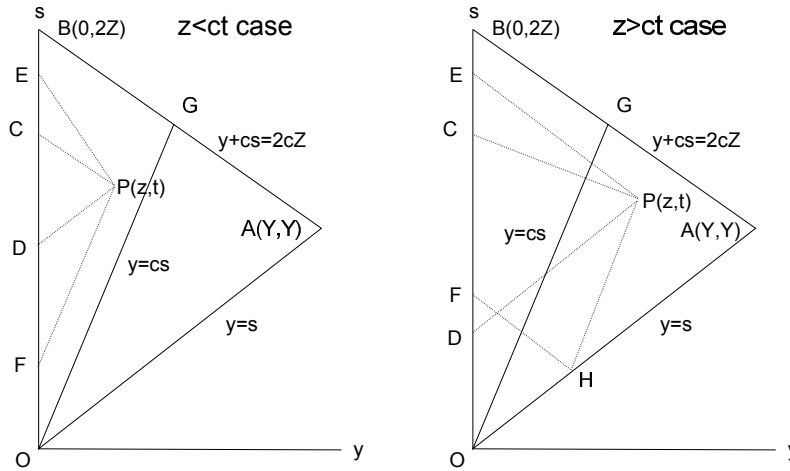


Figure 4.2: Leftward moving lines through $P(z, t)$ with slopes ± 1 and $\pm 1/c$

Again, the existence and uniqueness of a unique C^1 solution for (4.4a)-(4.4d) may be proved by standard arguments for Volterra equations or one may use the method in section 2.5 of [9]. Of course, because of the piecewise nature of the fourth integral equation (4.4d), some calculations are needed to confirm the C^1 regularity of h_4 and the matching conditions (4.3a)-(4.3b) will be required for the C^1 regularity at the origin. \square

To set up the fixed point problem later in this section, we give meaning to and prove the existence of the unique solution of (4.2a)-(4.2c) when $\beta(\cdot) \in L^2[0, Y]$. Further, we show that this solution has an L^2 trace on the line $t = z$.

Proposition 4.2. *If $\mathbf{a} \in C^1[0, 2Z]$ satisfies (4.3a), (4.3b), and $\mathbf{h} \in C^1(\tilde{\Omega})$ is the corresponding solution of (4.2a)-(4.2c), then the solution map \mathcal{S} and the solution trace map \mathcal{T}*

$$\begin{aligned} \mathcal{S} : C^1[0, Y] &\rightarrow C^1(\tilde{\Omega}) & \text{and} & & \mathcal{T} : C^1[0, Y] &\rightarrow C^1[0, Y] \\ \beta(\cdot) &\rightarrow \mathbf{h}(\cdot, \cdot) & & & \beta(\cdot) &\rightarrow h_3(z, z) \end{aligned}$$

are locally Lipschitz continuous with respect to the L^2 norms on the domain and the codomains⁴.

Proof. Suppose $\beta, \tilde{\beta} \in C^1[0, Y]$ and $\|\beta\|_{L^2[0, Y]}^2 \leq K$, $\|\tilde{\beta}\|_{L^2[0, Y]}^2 \leq K$ and WLOG we assume that $\|\mathbf{a}\|_{L^2[0, 2Z]} \leq K$. Let $\mathbf{h}, \tilde{\mathbf{h}} \in C^1(\tilde{\Omega})$ be the solutions of (4.2a) – (4.2c) corresponding to $\beta, \tilde{\beta}$.

Applying Lemma 3.1 to $\tilde{\mathbf{h}}$, from (3.4) we have

$$J(\tilde{\mathbf{h}}, z) \leq C J(\tilde{\mathbf{h}}, 0) =: C_0, \quad \forall z \in [0, Y], \quad (4.5)$$

where C_0 depends only on c, Z, K .

If we define $\mathbf{p} = \mathbf{h} - \tilde{\mathbf{h}}$ then \mathbf{p} satisfies $\mathcal{L}\mathbf{p} = (\beta - \tilde{\beta})B\tilde{\mathbf{h}}$ with $\mathbf{p}(0, t) = 0$ for $t \in [0, 2Z]$ and

$$p_3(z, z) = \frac{(c-1)^2}{(c+1)^2} p_4(z, z), \quad 0 \leq z \leq Y.$$

Hence, applying Lemma 3.1 to \mathbf{p} and taking $\lambda = 1$, $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$, from (3.2) we obtain (see Figure 3.1)

$$\begin{aligned} J(\mathbf{p}, z) + \int_{OC} p_3^2(y, y) dy &\leq \iint_{OCDB} |(\mathcal{L}\mathbf{p})(y, t)|^2 dx dt + \frac{1}{c\epsilon} \int_0^z (1 + 4|\beta(y)|) J(\mathbf{p}, y) dy \\ &\leq 4 \int_0^z (\beta - \tilde{\beta})^2(y) J(\tilde{\mathbf{h}}, y) dy + \frac{1}{c\epsilon} \int_0^z (1 + 4|\beta(y)|) J(\mathbf{p}, y) dy \\ &\leq 4C_0 \int_0^z (\beta - \tilde{\beta})^2(y) dy + \frac{1}{c\epsilon} \int_0^z (1 + 4|\beta(y)|) J(\mathbf{p}, y) dy \end{aligned}$$

where we used (4.5) in the last step. Hence, using Gronwall's inequality, we have

$$J(\mathbf{p}, z) + \int_{OC} p_3^2(y, y) dy \leq C \int_0^Y (\beta - \tilde{\beta})^2(y) dy$$

with the constant dependent only on c, Z, K . This is enough to prove the proposition. \square

⁴The set containing the range

Since C^1 is dense in L^2 and \mathcal{S} and \mathcal{T} are locally Lipschitz continuous in the L^2 norm, \mathcal{S} and \mathcal{T} have unique continuous extensions $\overline{\mathcal{S}}$ and $\overline{\mathcal{T}}$

$$\begin{aligned} \overline{\mathcal{S}} : L^2[0, Y] &\rightarrow L^2(\tilde{\Omega}) & \text{and} & & \overline{\mathcal{T}} : L^2[0, Y] &\rightarrow L^2[0, Y] \\ \beta(\cdot) &\rightarrow \mathbf{h}(\cdot, \cdot) & & & \beta(\cdot) &\rightarrow h_3(z, z). \end{aligned}$$

For $\beta \in L^2[0, Y]$, $\overline{\mathcal{S}}\beta$ is a candidate for a weak solution of (4.2a)-(4.2c) and $\overline{\mathcal{T}}\beta$ is the candidate for the trace of this solution on $t = z$. Of course, we must first define what we mean by a weak solution of (4.2a)-(4.2c).

For arbitrary $\mathbf{h}, \mathbf{n} \in C^1(\tilde{\Omega})$, from (3.1) and the divergence theorem (see Figure 3.1) we have

$$\iint_{\tilde{D}} (\mathcal{L}\mathbf{n})^T \mathbf{h} + \mathbf{n}^T \mathcal{L}\mathbf{h} \, dx \, dt = \int_{AB} \mathbf{n}^T (cI - A) \mathbf{h} \, dt - \int_{OA} \mathbf{n}^T (I + A) \mathbf{h} \, dt + \int_{OB} \mathbf{n}^T A \mathbf{h} \, dt.$$

Now $cI - A$ is a diagonal matrix with only the first, second and fourth diagonal entries being non-zero while $I + A$ is a diagonal matrix with only the first, third and fourth diagonal entries being non-zero. So keeping in mind (4.2b) the following seems an appropriate definition of a weak solution of (4.2a)-(4.2c).

Definition 4.1. For arbitrary $\beta \in L^2[0, Y]$ and arbitrary $\mathbf{a} \in C^1[0, 2Z]$ which satisfies (4.3a), (4.3b), we say that $\mathbf{h} \in L^2(\tilde{\Omega})$ is a weak solution of (4.2a)-(4.2c) if (see Figure 4.2)

$$\iint_{\tilde{D}} (\mathbf{n}_t - A\mathbf{n}_z - \beta B\mathbf{n})^T \mathbf{h} \, dx \, dt = \int_{OB} \mathbf{n}^T A \mathbf{a} \, dt \quad (4.6)$$

for all \mathbf{n} in

$$\Lambda := \{\mathbf{n} \in C^1(\tilde{\Omega}) \mid n_1 = 0, (1 - c)n_3 + (1 + c)n_4 = 0 \text{ on } OA, n_1 = n_2 = n_4 = 0 \text{ on } AB\}.$$

We now show the uniqueness and existence of the weak solution of (4.2a)-(4.2c).

Proposition 4.3 (Weak solution of the sideways CBVP). For any $\beta \in L^2[0, Y]$ and arbitrary $\mathbf{a} \in C^1[0, 2Z]$ which satisfies (4.3a), (4.3b), $\overline{\mathcal{S}}\beta$ is the unique weak solution of (4.2a)-(4.2c). Further, $\overline{\mathcal{S}}\beta$ has an L^2 trace on $t = z$ which is $\overline{\mathcal{T}}\beta$.

Proof. Given $\beta \in L^2[0, Y]$, we can find a sequence $\beta_k \in C^1[0, Y]$ which converges to β in the L^2 norm. Let $\mathbf{h}_k = \mathcal{S}\beta_k$ be the C^1 solution of (4.2a)-(4.2c). Since \mathcal{S} is locally Lipschitz, \mathbf{h}_k will be a Cauchy sequence in $L^2(\tilde{\Omega})$ and hence has a limit $\mathbf{h} \in L^2(\tilde{\Omega})$; in fact this \mathbf{h} defines $\overline{\mathcal{S}}\beta$. Now \mathbf{h}_k, β_k satisfy (4.6) for all $\mathbf{n} \in \Lambda$, so from the L^2 convergence it is clear that (4.6) will hold for the L^2 limit of β_k and \mathbf{h}_k . Hence $\overline{\mathcal{S}}\beta$ is a weak solution of (4.2a)-(4.2c). Further, the construction of $\overline{\mathcal{S}}$ and $\overline{\mathcal{T}}$ shows that $\overline{\mathcal{T}}\beta$ is the trace of this solution on $t = z$.

It remains to prove the uniqueness of the weak solution. To show uniqueness it is enough to show that if $\mathbf{h} \in L^2(\tilde{\Omega})$, $\beta \in L^2[0, Y]$ and

$$\iint_{\tilde{D}} (\mathbf{n}_t - A\mathbf{n}_z - \beta B\mathbf{n})^T \mathbf{h} \, dz \, dt = 0, \quad \forall \mathbf{n} \in \Lambda$$

then $\mathbf{h} = 0$.

From the density of C^1 in L^2 , we can find sequences $\beta_k \in C^1[0, Y]$ and $\mathbf{h}^k \in C^1(\tilde{\Omega})$ whose L^2 limits are β and \mathbf{h} respectively. We show below that we can find $\mathbf{n}^k \in \Lambda$ such that $\mathbf{n}_t^k - A\mathbf{n}_z^k - \beta_k B\mathbf{n}^k = \mathbf{h}^k$ in $\tilde{\Omega}$ and further $\sup_{z \in [0, Y]} J(\mathbf{n}^k, z)$ is bounded above by a constant independent of k . Assuming this for the moment we have

$$\begin{aligned} 0 &= \iint_{\tilde{D}} (\mathbf{n}_t^k - A\mathbf{n}_z^k - \beta B\mathbf{n}^k)^T \mathbf{h} \, dz \, dt \\ &= \iint_{\tilde{D}} (\mathbf{n}_t^k - A\mathbf{n}_z^k - \beta_k B\mathbf{n}^k)^T \mathbf{h} \, dz \, dt + \iint_{\tilde{D}} (\beta - \beta_k) (B\mathbf{n}^k)^T \mathbf{h} \, dz \, dt \\ &= \iint_{\tilde{D}} \mathbf{h}^T \mathbf{h}^k \, dz \, dt + \iint_{\tilde{D}} (\beta - \beta_k) (B\mathbf{n}^k)^T \mathbf{h} \, dz \, dt. \end{aligned} \tag{4.7}$$

Now (see Figure 3.1)

$$\begin{aligned} \left| \iint_{\tilde{D}} (\beta - \beta_k) (B\mathbf{n}^k)^T \mathbf{h} \, dz \, dt \right| &\leq \int_0^Y |(\beta - \beta_k)(z)| \int_{CD} \|\mathbf{n}^k(z, t)\| \|\mathbf{h}(z, t)\| \, dt \, dz \\ &\leq \int_0^Y |(\beta - \beta_k)(z)| \sqrt{J(\mathbf{n}^k, z) J(\mathbf{h}, z)} \, dz \\ &\leq \|\beta - \beta_k\|_{L^2[0, Y]} \left(\int_0^Y J(\mathbf{n}^k, z) J(\mathbf{h}, z) \, dz \right)^{1/2} \\ &\leq \|\beta - \beta_k\|_{L^2[0, Y]} \left(\int_0^Y J(\mathbf{h}, z) \, dz \right)^{1/2} \\ &= \|\beta - \beta_k\|_{L^2[0, Y]} \|\mathbf{h}\|_{L^2(\tilde{\Omega})} \end{aligned}$$

which approaches 0 as $k \rightarrow \infty$. Hence, from (4.7), taking the limit as $k \rightarrow \infty$, we obtain $\|\mathbf{h}\|_{L^2(\tilde{\Omega})}^2 = 0$ and hence $\mathbf{h} = 0$.

So it remains to show that if $\beta \in C^1[0, Y]$, then for any $\mathbf{F}(z, t) \in C^1(\tilde{\Omega})$ there is an $\mathbf{n} \in C^1(\tilde{\Omega})$ such that (see Figure 4.3)

$$\mathbf{n}_t - A\mathbf{n}_z - \beta B\mathbf{n} = \mathbf{F} \text{ in } \tilde{\Omega}, \tag{4.8a}$$

$$n_1 = 0, \quad (1 - c)n_3 + (1 + c)n_4 = 0 \text{ on } OA, \tag{4.8b}$$

$$n_1 = n_2 = n_4 = 0 \text{ on } AB; \tag{4.8c}$$

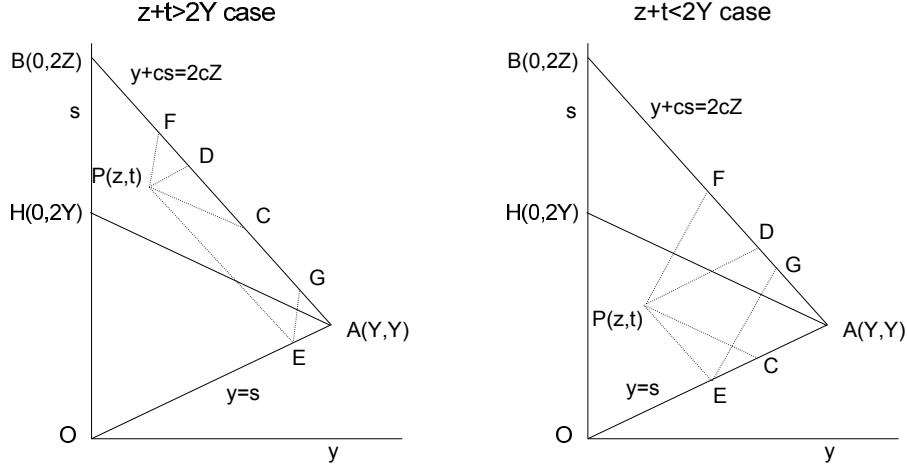


Figure 4.3: Rightward moving lines through $P(z, t)$ with slopes ± 1 and $\pm 1/c$

further $\sup_{z \in [0, Y]} J(\mathbf{n}, z)$ is bounded above by a constant determined only by $\|\mathbf{F}\|_{L^2(\tilde{\Omega})}$, $\|\beta\|_{L^2[0, Y]}$, Z and c .

Define $\mathbf{r}(\mathbf{n}, z, t) := \beta(z)B\mathbf{n}(z, t) + \mathbf{F}(z, t)$, and pick an arbitrary point $P(z, t) \in \tilde{\Omega}$. Integrating (4.8a) along the characteristics and using the boundary conditions, we obtain the system of integral equations

$$n_1(z, t) = - \int_z^{y_C} r_1(\mathbf{n}, y, z + t - y) dy, \quad (4.9a)$$

$$n_2(z, t) = - \int_z^{y_D} r_2(\mathbf{n}, y, y + t - z) dy, \quad (4.9b)$$

$$n_3(z, t) = -\frac{1}{c} \int_z^{y_E} r_3\left(\mathbf{n}, y, \frac{z + ct - y}{c}\right) dy + \frac{1+c}{c(1-c)} \int_{y_E}^{y_G} r_4\left(\mathbf{n}, y, \frac{y + cy_E - y_E}{c}\right) dy, \quad (4.9c)$$

$$n_4(z, t) = \frac{1}{c} \int_z^{y_F} r_4\left(\mathbf{n}, y, \frac{y + ct - z}{c}\right) dy. \quad (4.9d)$$

The existence of a C^1 solution of this system of integral equations holds by the usual argument. The upper bound on $J(\mathbf{n}, z)$ may be obtained by using arguments similar to those used in the proof of Lemma 3.1. The only difference is that the identity (3.10) must be integrated over the region CAD (see Figure 3.1) instead of the region OCDB and one should now choose $\epsilon = (1+c)^4/(1-c)^3$.

□

4.2 Local Reconstruction

Suppose $\beta \in \dot{C}^1[0, 2Z]$ and $\mathbf{m}(z, t)$ is the C^1 solution of (1.4a) - (1.4c). Given $m_1(0, t), m_3(0, t)$ on $[0, 2Z]$, our goal is to reconstruct β on $[0, Y]$. The reconstruction will occur piece by piece, first over an interval $[0, \delta]$, then over $[\delta, 2\delta]$, then $[2\delta, 3\delta]$, and so on, with the $\delta > 0$ determined by the value of $m_1(0, t), m_3(0, t)$ on $[0, 2Z]$. The crux of the global reconstruction is a local reconstruction as described next. Suppose $0 \leq X < Y$; given $\beta(X)$ and $\mathbf{m}(X, t)$ for

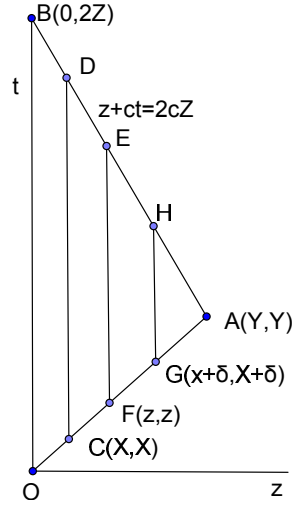


Figure 4.4: Local Reconstruction

all $t \in [X, (2cZ - X)/c]$ (that is given \mathbf{m} on CD - see figure 4.4), we reconstruct β on the interval $[X, X + \delta]$ for some $\delta > 0$.

For arbitrary $X \in [0, Y]$ and $\delta > 0$ such that $X + \delta \leq Y$, define

$$\tilde{D}_{X,\delta} := \{(z, t) \mid (z, t) \in \tilde{D}, X \leq z \leq X + \delta\}.$$

Further, for any $K_X > 0$, define the complete metric space (in the L^2 norm)

$$\overline{\mathcal{B}}_X := \{\beta \in L^2[X, X + \delta] \mid \|\beta\|_{L^2[X, X + \delta]}^2 \leq K_X\},$$

which, for an arbitrary fixed real number β_* , has a dense subset

$$\mathcal{B}_X := \{\beta \in C^1[X, X + \delta] \mid \|\beta\|_{L^2[X, X + \delta]}^2 \leq K_X, \beta(X) = \beta_*\}.$$

For an arbitrary $\mathbf{a}(\cdot) \in C^1[X, (2cZ - X)/c]$, consider the CBVP

$$\mathbf{h}_t = A\mathbf{h}_z + \beta B\mathbf{h}, \quad \text{in } \tilde{D}_{X,\delta}, \quad (4.10a)$$

$$h_3(z, z) = \frac{(c-1)^2}{(c+1)^2} h_4(z, z), \quad X \leq z \leq X + \delta, \quad (4.10b)$$

$$\mathbf{h}(X, t) = \mathbf{a}(t), \quad t \in [X, (2cZ - X)/c]. \quad (4.10c)$$

For $\beta \in C^1[X, X + \delta]$, just as in Proposition 4.1, one can verify that the matching conditions on $\mathbf{a}(\cdot)$ needed for a C^1 solution of (4.10a)-(4.10c) are

$$(c+1)^2 a_3(X) = (1-c)^2 a_4(X), \quad (4.11a)$$

$$(c+1)^2 ((1+c)a'_3(X) - \beta(X)(B\mathbf{a})_3(X)) = (c-1)^2 ((c-1)a'_4(X) + \beta(X)(B\mathbf{a})_4(X)). \quad (4.11b)$$

Here is the important local reconstruction step.

Proposition 4.4. *Let $0 \leq X \leq Y$, $\beta_* \in \mathbb{R}$ and $\mathbf{a} \in C^1[X, (2cZ - X)/c]$ such that (4.11a), (4.11b) hold. There exists a $\delta > 0$, $K_X > 0$ and a unique $\beta \in \bar{\mathcal{B}}_X$ with*

$$h_3(z, z) = \frac{c-1}{2(c+1)} \beta(z), \quad X \leq z \leq X + \delta, \quad (4.12)$$

where $\mathbf{h}(z, t) \in L^2(\tilde{D}_{Z,\delta})$ is the unique weak solution of (4.10a)-(4.10c). Actually, it is sufficient to choose any $K_X > 0$ and $\delta > 0$ so that

$$K_X \geq \frac{8(1+c)^2}{(1-c)^2} J_X, \quad \delta \leq \min\left(Y - X, \frac{c^2 \epsilon^2}{256 K_X}\right), \quad (4.13)$$

where $J_X = \|a_1\|_{L^2}^2 + \|a_2\|_{L^2}^2 + c\|a_3\|_{L^2}^2 + c\epsilon\|a_4\|_{L^2}^2$ and $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$.

Proof. If $\beta \in C^1[X, X + \delta]$ then (extending β arbitrarily to a function in $C^1[X, Y]$), from Proposition 4.1, (4.10a)-(4.10c) has a unique solution $\mathbf{h} \in C^1[\tilde{D}_{X,\delta}]$. So we may define the map

$$Q : \beta \rightarrow \frac{2(c+1)}{c-1} h_3(z, z)$$

and our goal is to find a fixed point for this map. We do so by setting up Q as a contraction map on a complete metric space.

First we show that for appropriate $\delta > 0$ and $K_X > 0$, if $\beta \in \mathcal{B}_X$ then $Q\beta \in \mathcal{B}_X$. Given $\beta \in \mathcal{B}_X$, let \mathbf{h} be the corresponding unique C^1 solution of (4.10a)-(4.10c). From Lemma 3.1 (using $\mathcal{L}\mathbf{h} = 0$ and letting $\lambda \rightarrow \infty$) we have (see Figure 4.4)

$$\begin{aligned} J(\mathbf{h}, z) + \int_{CF} (2h_1^2 + (1+c)h_3^2 - \epsilon(1-c)h_4^2) dy \\ \leq J_X + \frac{4}{c\epsilon} \int_X^z |\beta(y)| J(\mathbf{h}, y) dy, \quad z \in [X, X + \delta]. \end{aligned}$$

If we take $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$ and use the characteristic condition (4.10b) then

$$\int_{CF} ((1+c)h_3^2 - \epsilon(1-c)h_4^2) dt = \int_{CF} h_3^2 dt$$

and hence

$$J(\mathbf{h}, z) + \int_{CF} h_3^2 \leq J_X + \frac{4}{c\epsilon} \int_X^z |\beta(y)| J(\mathbf{h}, y) dy, \quad z \in [X, X + \delta]; \quad (4.14)$$

so from Gronwall's inequality (for use later)

$$J(\mathbf{h}, z) \leq e^{4\sqrt{K_X\delta}/(c\epsilon)} J_X, \quad z \in [X, X + \delta]. \quad (4.15)$$

Define $J^*(\mathbf{h}) := \max_{z \in [X, X+\delta]} J(\mathbf{h}, z)$; then from (4.14) we have

$$J^*(\mathbf{h}) + \int_{CG} h_3^2 dz \leq 2J_X + \frac{8}{c\epsilon} \int_X^{X+\delta} |\beta(y)| J^*(\mathbf{h}) dy \leq 2J_X + \frac{8\sqrt{\delta K_X}}{c\epsilon} J^*(\mathbf{h}),$$

which implies that

$$\left(1 - \frac{8\sqrt{\delta K_X}}{c\epsilon}\right) J^*(\mathbf{h}) + \int_{CG} h_3^2 dz \leq 2J_X.$$

So if $\delta \leq \frac{c^2\epsilon^2}{64K_X}$, we have

$$\|Q\beta\|_{L^2[X, X+\delta]}^2 = \frac{4(1+c)^2}{(1-c)^2} \int_X^{X+\delta} h_3(z, z)^2 dz \leq \frac{8(1+c)^2}{(1-c)^2} J_X \leq K_X,$$

if we choose $K_X \geq \frac{8(1+c)^2}{(1-c)^2} J_X$.

Summarizing, if we chose

$$\epsilon = \frac{c(1-c)^3}{(1+c)^4}, \quad K_X \geq \frac{8(1+c)^2}{(1-c)^2} J_X, \quad \delta \leq \frac{c^2\epsilon^2}{64K_X}, \quad (4.16)$$

then we have a map

$$Q : \mathcal{B}_X \mapsto \mathcal{B}_X, \\ (Q\beta)(z) = \frac{2(c+1)}{c-1} h_3(z, z).$$

We now show that Q is a contraction if $\delta > 0$ is small enough.

Suppose $\beta, \tilde{\beta} \in \mathcal{B}_X$ and let $\mathbf{h}, \tilde{\mathbf{h}}$ be the corresponding unique C^1 solutions of (4.10a) – (4.10c). Define $\mathbf{p} := \mathbf{h} - \tilde{\mathbf{h}}$; then $\mathcal{L}(\mathbf{p}) = (\beta - \tilde{\beta})B\tilde{\mathbf{h}}$, $\mathbf{p} = 0$ on $z = X$, and p_3, p_4 satisfy

(4.10b) on $z = t$. Apply Lemma 3.1 but on the interval $[X, X + \delta]$ instead of the interval $[0, Y]$, and choose $\epsilon = \frac{c(1-c)^3}{(1+c)^4}$. Noting that (as vector norms) $\|Bh\|^2 \leq 4\|h\|^2$ and using (4.15) for \tilde{h} , we obtain

$$\begin{aligned} J^*(\mathbf{p}) + \int_{CG} p_3^2 dy &\leq \frac{8\lambda}{c\epsilon} \int_X^{X+\delta} (\beta - \tilde{\beta})^2(y) J(\tilde{\mathbf{h}}, y) dy + \frac{2}{c\epsilon} \int_X^{X+\delta} \left(4|\beta(y)| + \frac{1}{\lambda} \right) J^*(\mathbf{p}) dy \\ &\leq \frac{8\lambda e^{4\sqrt{K_X\delta}/(c\epsilon)} J_X}{c\epsilon} \int_X^{X+\delta} (\beta - \tilde{\beta})^2(y) dy + \frac{2\delta/\lambda + 8\sqrt{\delta K_X}}{c\epsilon} J^*(\mathbf{p}) \end{aligned}$$

which implies that

$$\left(1 - \frac{2\delta/\lambda + 8\sqrt{\delta K_X}}{c\epsilon} \right) J^*(\mathbf{p}) + \int_{CG} p_3^2 dy \leq \frac{8\lambda e^{4\sqrt{K_X\delta}/(c\epsilon)} J_X}{c\epsilon} \int_X^{X+\delta} (\beta - \tilde{\beta})^2(y) dy.$$

So choosing

$$\delta \leq \min \left(\frac{c\lambda\epsilon}{4}, \frac{c^2\epsilon^2}{256K_X} \right), \quad (4.17)$$

we have

$$\int_{CG} p_3^2 dy \leq \frac{8\lambda e^{4\sqrt{K_X\delta}/(c\epsilon)} J_X}{c\epsilon} \int_X^{X+\delta} (\beta - \tilde{\beta})^2(y) dy$$

which implies

$$\|Q\beta - Q\tilde{\beta}\|_{L^2[X, X+\delta]}^2 \leq \sigma \|\beta - \tilde{\beta}\|_{L^2[X, X+\delta]}^2 \quad (4.18)$$

where

$$\sigma = \frac{4(1+c)^2}{(1-c)^2} \frac{8\lambda e^{4\sqrt{K_X\delta}/(c\epsilon)} J_X}{c\epsilon}.$$

The constraints on K_X, δ and λ are given by (4.16), (4.17). We take

$$\lambda = \frac{c\epsilon(1-c)^2}{64(1+c)^2 J_X} e^{-4\sqrt{K_X\delta}/(c\epsilon)} \quad (4.19)$$

then $\sigma = 1/2$; so we now have to choose $\delta > 0$ small enough so that (4.16) and (4.19) imply (4.17). Some calculations⁵ will show that choosing any δ with

$$\delta \leq \min \left(Y - X, \frac{c^2\epsilon^2}{256K_X} \right)$$

⁵ Now from the last inequality in (4.16) we have $\frac{4\sqrt{K_X\delta}}{c\epsilon} \leq \frac{1}{2}$; hence, using (4.19) and the second inequality in (4.16), we have

$$\lambda \geq \frac{c\epsilon(1-c)^2}{64(1+c)^2 J_X} e^{-1/2} \geq \frac{c\epsilon(1-c)^2}{8(1-c)^2 K_X} e^{-1/2} \geq \frac{c\epsilon}{24K_X}.$$

So $\frac{c\lambda\epsilon}{4} \geq \frac{c^2\epsilon^2}{96K_X}$ and hence (4.17) holds if $\delta \leq \frac{c^2\epsilon^2}{256K_X}$.

will satisfy (4.16), (4.17). Hence choosing K_X and $\delta > 0$ which satisfy (4.13), we have shown that Q is a contraction map with $\sigma = 1/2$ in (4.18).

So Q has an extension \overline{Q} to the complete metric space $\overline{\mathcal{B}}_X$, namely

$$\begin{aligned}\overline{Q} : \overline{\mathcal{B}}_X &\mapsto \overline{\mathcal{B}}_X, \\ (\overline{Q}\beta)(z) &= \frac{2(c+1)}{c-1} h_3(z, z)\end{aligned}$$

where $\mathbf{h} = \overline{\mathcal{S}}\beta$ and \mathbf{h} has an L^2 trace on $z = t$ (because of Proposition 4.3). Further, \overline{Q} will also be a contraction map, and hence have a unique fixed point, which may be obtained by an algorithm.

□

4.3 Global reconstruction

We defined the forward map \mathcal{F}

$$\begin{aligned}\mathcal{F} : \dot{C}^1[0, Z] &\mapsto C^1[0, 2Z] \times C^1[0, 2Z], \\ (\mathcal{F}\beta)(z) &= [m_1(0, t), m_3(0, t)]\end{aligned}$$

where \mathbf{m} is the solution of (1.4a) – (1.4c) and we have shown in Theorem 1.2 that \mathcal{F} is injective.

Since $[\phi(\cdot), \psi(\cdot)]$ are in the range of \mathcal{F} , there is a unique (unknown) $\beta(\cdot) \in \dot{C}^1[0, Z]$ and a corresponding unique (unknown) C^1 solution \mathbf{m} of (1.4a)-(1.4c) such that $m_1(0, t) = \phi(0, t)$, $m_3(0, t) = \psi(0, t)$, $t \in [0, 2Z]$. As per the hypothesis, we also assume that $\|\beta\|_{L^2[0, Y]}^2 \leq K$ for some known $K \geq 0$, for this unique unknown β .

Applying Proposition 4.4 with $X = 0$, $\beta_* = 0$, $\mathbf{a} = [\phi, 0, \psi, 0]$ (note that the C^1 matching conditions (4.11a), (4.11b) hold because we already know the existence of a C^1 solution, namely \mathbf{m}), we can find a $\delta_0 > 0$, $K_0 > 0$ and a unique $\beta(\cdot) \in L^2[0, \delta]$ such that (4.12) holds where $\mathbf{h}(z, t)$ is the solution of (4.10a)-(4.10c). This β must be same as $\mathcal{F}^{-1}[\phi, \psi]$ because the \mathbf{m} corresponding to $\mathcal{F}^{-1}[\phi, \psi]$ already satisfies (4.10a)-(4.10c). Since Proposition 4.4 was constructive, we have recovered β on the interval $[0, \delta_0]$. Further, from Prop 4.1 applied to the interval $[0, \delta_0]$ with $\mathbf{a} = [\phi, 0, \psi, 0]$ we can construct the unique C^1 solution of (4.2a)-(4.2c) on the region $\tilde{D}_{X, \delta}$, which is the \mathbf{m} corresponding to the $\beta = \mathcal{F}^{-1}[\phi, \psi]$. Hence we now also have $\mathbf{m}(\delta_0, \cdot)$ on the interval $[\delta_0, (2cZ - \delta_0)/c]$ as well as $\beta(\delta_0)$.

Now we show the general step. Suppose, for some $X > 0$ we are given $\mathbf{m}(X, \cdot)$ on the interval $[X, (2cZ - X)/c]$ as well as $\beta(X)$. Then the C^1 matching conditions are automatically satisfied and hence an application of Proposition 4.4 with $\mathbf{a}(\cdot) = \mathbf{m}(X, \cdot)$, there exists a $\delta > 0$ such that we can recover $\beta(\cdot)$ on the interval $[X, X + \delta]$. Then repeating the argument in the previous paragraph we can calculate $\mathbf{m}(X + \delta, t)$ for all $t \in [X + \delta, (2cZ - X - \delta)/c]$.

We can then apply this process repeatedly. To show that this process will end in a finite number of steps, we need to obtain a lower bound on the step size δ guaranteed by Proposition 4.4. Let $\beta = \mathcal{F}^{-1}[\phi, \psi]$ and \mathbf{m} the corresponding solution of (1.4a)-(1.4c); note that β and \mathbf{m} are unknown but $\mathbf{m}(0, t) = [\phi(t), 0, \psi(t), 0]$, $t \in [0, 2Z]$ is given to us and $\|\beta\|_{L^2[0, Y]}^2 \leq K$ for some known $K \geq 0$. Applying Lemma 3.1 for this β and \mathbf{m} , (3.4) implies that

$$J(\mathbf{m}, z) \leq e^{4\sqrt{KY}/(c\epsilon)}, \quad z \in [0, Y] \quad (4.20)$$

where $\epsilon = c(1 - c)^3/(1 + c)^4$. Now at each iteration step we applied Proposition 4.4 with $\mathbf{a} = \mathbf{m}$, so from (4.20)

$$J_X = J(\mathbf{m}, X) \leq e^{4\sqrt{KY}/(c\epsilon)}$$

and hence if we take

$$K_X = \frac{8(1 + c)^2}{(1 - c)^2} J_X$$

then

$$\frac{c^2\epsilon^2}{256K_X} = \frac{c^2(1 - c)^2\epsilon^2}{2048(1 + c)^2J_X} \geq \frac{c^2(1 - c)^2\epsilon^2}{2048(1 + c)^2} e^{-4\sqrt{KY}/(c\epsilon)} =: \delta_*.$$

So at every step we can choose a step size δ_* independent of X , except for the last step when the step size will be $\min(Y - X, \delta_*)$.

5 Numerical reconstruction

We now show the results from a numerical implementation of the scheme suggested by the proof of Theorem 1.3. The proof involved the construction of a fixed point for a contraction map Q ; the fixed point is the limit of the sequence β_n where β_0 is chosen arbitrarily and $\beta_{n+1} = Q\beta_n$.

The data for the inverse problem, for the chosen β , was generated by solving the CBVP (1.4a)-(1.4c) using the Crank-Nicolson method with interpolation to solve the ODE along the characteristics. The solution of the inverse problem requires solving the sideways CBVP (4.1a)-(4.1c), again using the Crank-Nicolson method with interpolation to solve the ODE

along the characteristics. In the examples below we used

$$c = 0.5, \quad Z = \frac{\pi}{2}$$

and N represents the number of subdivisions of $[0, Z]$. If β is the exact value and β_{app} the numerical approximation from our inversion then we plot the L^2 error E_2 and the relative L^∞ error E_∞ to judge the effectiveness of the algorithm where

$$E_2 = \left(\frac{Z}{N} \sum_{i=1}^N (\beta - \beta_{app})^2(z) \right)^{1/2}, \quad E_\infty = \max_{\beta(z_i) \neq 0} \left| \frac{(\beta - \beta_{app})(z_i)}{\beta(z_i)} \right|.$$

In the examples below, the calculated β and the exact β are very close compared to the scale so we see only one graph even though we have drawn two.

We apply the inversion scheme to four examples and we start with a simple example with just a little bit of oscillation.

Example 1 Here $\beta(z) = 3z^2 \cos(10z) \log(z + 1)$, we use an initial guess $\beta_0(z) = z$ and $N = 2^9$. The iterations converged in 17 steps. Figures 5.2 and 5.1 show the accuracy of our reconstruction.

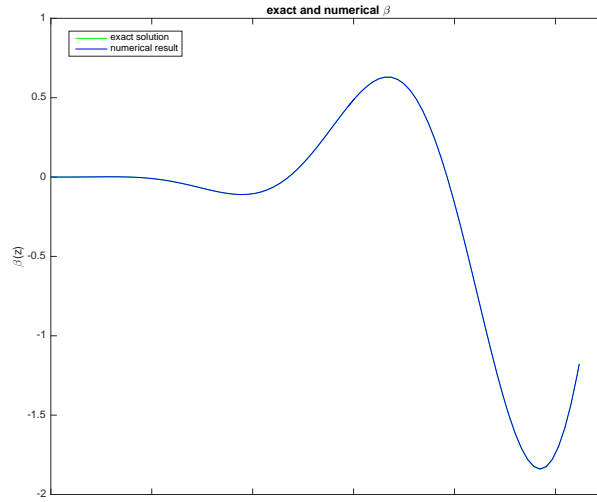


Figure 5.1: Comparing exact β with reconstructed β

Example 2 Here $\beta(z) = z \sin(100z) \log(z + 1)$, an initial guess $\beta_0(z) = z$ and $N = 2^6, 2^7, \dots, 2^{11}$. In all cases, the iterations converged in 17 steps and Figures 5.3 and 5.4 reflect the accuracy of our reconstruction.

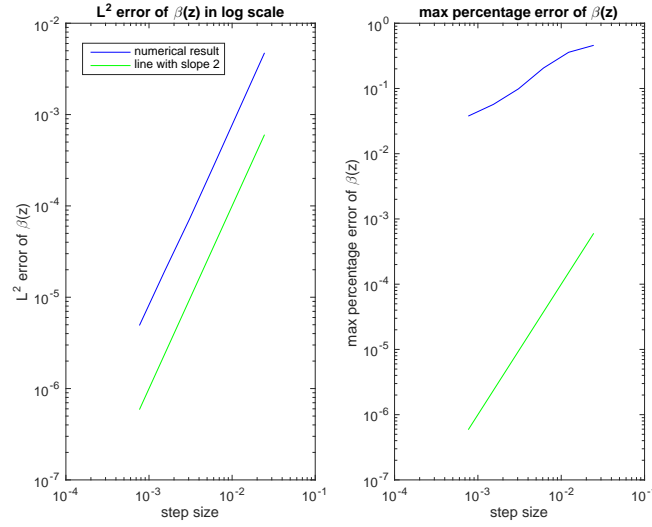


Figure 5.2: L^2 error and relative L^∞ error

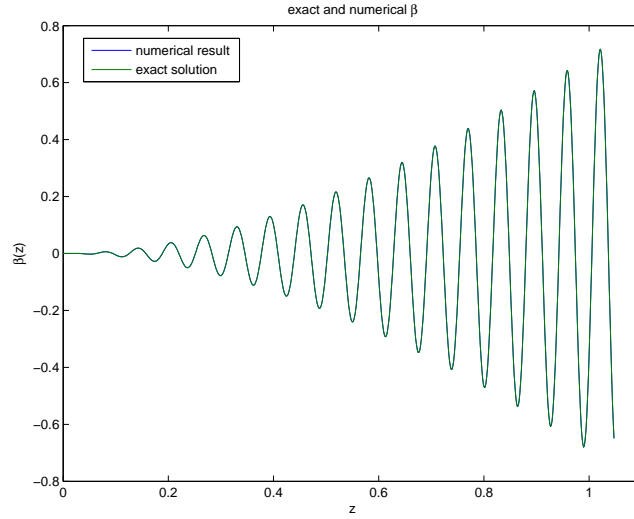


Figure 5.3: Comparing exact β with reconstructed β

Example 3 Here $\beta(z) = 9z^2 \cos(100z) \log(z + 1)$, an initial guess $\beta_0(z) = z$, and $N = 2^6, 2^7, \dots, 2^{11}$. In all cases, the iterations converged in 14 steps and Figures 5.5 and 5.6 reflect the accuracy of our reconstruction.

Example 4 Here $\beta(z) = z \sin(100z)e^{az}$ where a is an integer, an initial guess $\beta_0(z) = z$. For the algorithm to converge N had to be increased as a increased - see Table 5.1.

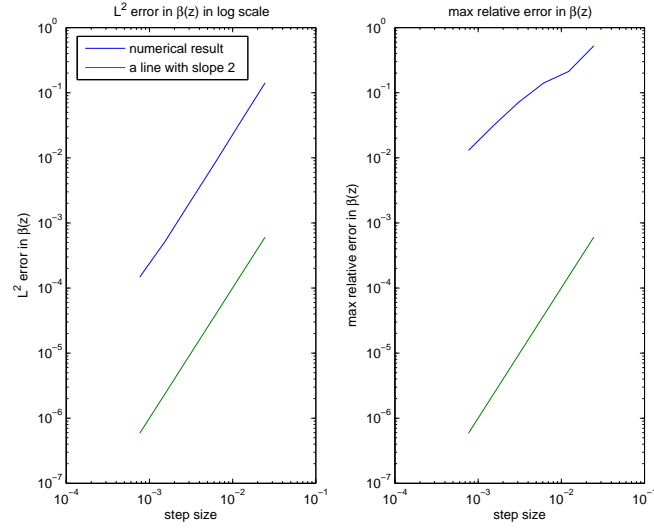


Figure 5.4: L^2 error and relative L^∞ error

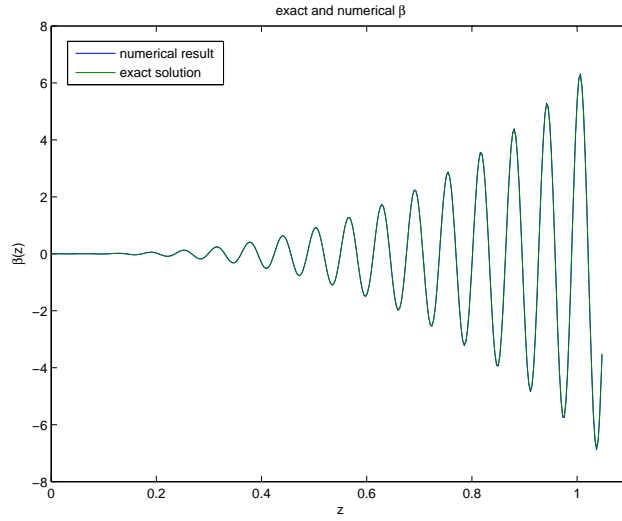


Figure 5.5: Comparing exact β with reconstructed β

a	3	4	4	5	5	6
N	2^6	2^7	2^8	2^9	2^{10}	2^{11}

Table 5.1: N value for the algorithm to converge

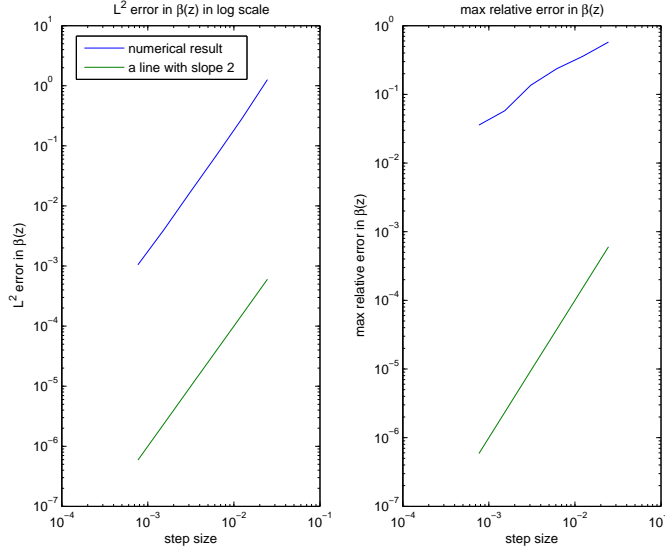


Figure 5.6: L^2 error and relative L^∞ error

6 Derivation of the model

During the 2000 Mathematical Problems in Industry workshop at the University of Delaware, Greg Luther, then of Corning Inc., proposed the problem of modeling the twist in a birefringent optical fiber and determining this twist from the response of the fiber, measured at one end of the fiber, to an impulsive source applied at the same end of the fiber. He suggested [7], [8] as possible sources for information. A few months after the workshop, a model was proposed in [10]. Since this derivation is not readily available, it is included here.

Consider an optical fiber stretching along the z axis and let $\mathbf{E}(z, t)$, $\mathbf{P}(z, t)$ be the electric field and the polarization at the point z units away from the left end of the fiber; then \mathbf{E} and \mathbf{P} obey Maxwell's equations

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \frac{1}{c_0^2} \mathbf{E}_{tt} + \frac{1}{\epsilon_0 c_0^2} \mathbf{P}_{tt} \quad (6.1)$$

where c_0 is the speed of light in vacuum and ϵ_0 is the permittivity of free space. Assume that \mathbf{E} and \mathbf{P} have no component along the fiber; since \mathbf{E} and \mathbf{P} depend only on z and t , (6.1) reduces to

$$\mathbf{E}_{zz} = \frac{1}{c_0^2} \mathbf{E}_{tt} + \frac{1}{\epsilon_0 c_0^2} \mathbf{P}_{tt}. \quad (6.2)$$

At every point in the fiber, there are two unit orthogonal vectors $\mathbf{v}_1(z)$ and $\mathbf{v}_2(z)$ perpendicular to the fiber, which represent the polarization directions of the two channels in the

fiber. As the fiber twists along its length, the polarization directions change. Since \mathbf{v}_1 and \mathbf{v}_2 are orthogonal unit vectors in a plane perpendicular to the fiber, $d\mathbf{v}_1/dz$ is orthogonal to $\mathbf{v}_1(z)$ and hence $\frac{d\mathbf{v}_1}{dz} = \beta(z)\mathbf{v}_2$ for some real valued function $\beta(z)$ and one may then show that $\frac{d\mathbf{v}_2}{dz} = -\beta\mathbf{v}_1$.

Since $\mathbf{E}(z)$ has no component along the fiber, we may write $\mathbf{E} = E_1\mathbf{v}_1 + E_2\mathbf{v}_2$. Further we assume that the polarization vector \mathbf{P} is related to the electric field \mathbf{E} via

$$\mathbf{P} = \epsilon_0(\alpha_1 E_1 \mathbf{v}_1 + \alpha_2 E_2 \mathbf{v}_2)$$

where α_1, α_2 are real constants. Substituting these representations of \mathbf{E} and \mathbf{P} into (6.2), using the relations for the derivatives of \mathbf{v}_1 and \mathbf{v}_2 , and matching the \mathbf{v}_1 and \mathbf{v}_2 components we obtain

$$(E_{1z} - \beta E_2)_z - \beta(E_{2z} + \beta E_1) = \frac{1}{c_1^2} E_{1tt}, \quad (6.3a)$$

$$(E_{2z} + \beta E_1)_z + \beta(E_{1z} - \beta E_2) = \frac{1}{c_2^2} E_{2tt} \quad (6.3b)$$

where it is assumed that $1 + \alpha_i > 0$ and we define $c_i = \frac{c_0}{\sqrt{1 + \alpha_i}}$.

The second order hyperbolic system of equation (6.3a), (6.3b) has two speeds of propagation c_1, c_2 and E_1, E_2 are, respectively, the waves propagating at these speeds. We rewrite this system as a first order system where we distinguish between the right and left moving components of these waves. If we define $\mathbf{M} = [M_1, M_2, M_3, M_4]^T$ where

$$2M_1 = E_{1z} - \beta E_2 + \frac{1}{c_1} E_{1t}, \quad 2M_2 = E_{1z} - \beta E_2 - \frac{1}{c_1} E_{1t}, \quad (6.4a)$$

$$2M_3 = E_{2z} + \beta E_1 + \frac{1}{c_2} E_{2t}, \quad 2M_4 = E_{2z} + \beta E_1 - \frac{1}{c_2} E_{2t} \quad (6.4b)$$

then one may verify that $\mathbf{M}(z, t)$ satisfies (1.1a); here, WLOG (because of scaling), for convenience we have assumed that the faster speed $c_1 = 1$ and the smaller speed $c_2 = c$.

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